Image of the Spectral Measure of a Jacobi Field and the Corresponding Operators

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Abstract

By definition, a Jacobi field $J = (J(\phi))_{\phi \in H_+}$ is a family of commuting self-adjoint three-diagonal operators in the Fock space $\mathcal{F}(H)$. The operators $J(\phi)$ are indexed by the vectors of a real Hilbert space H_+. The spectral measure ρ of the field J is defined on the space H_- of functionals over H_+. The image of the measure ρ under a mapping $K^+ : T_- \to H_-$ is a probability measure ρ_K on T_-. We obtain a family J_K of operators whose spectral measure is equal to ρ_K. We also obtain the chaotic decomposition for the space $L^2(T_-, d\rho_K)$.

1   Introduction

Consider a real Hilbert space H and the corresponding symmetric Fock space

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H). \qquad (1.1)$$

Let

$$H_- \supset H \supset H_+$$

be a rigging of H with the quasinuclear embedding $H_+ \hookrightarrow H$. Consider a Jacobi field $J = (\tilde{J}(\phi))_{\phi \in H_+}$ in the space $\mathcal{F}(H)$. By definition, a Jacobi field is a family of commuting selfadjoint operators which have a three-diagonal structure with respect to the decomposition (1.1). These operators are assumed to linearly and continuously depend on the indexing parameter $\phi \in H_+$. The concept of a Jacobi field was studied in [7], [17], [1], [2], [3], and [4].

The above-mentioned papers provide the expansion of the Jacobi field J in generalized joint eigenvectors. The corresponding Fourier transform appears to be a unitary operator between the Fock space $\mathcal{F}(H)$ and the space $L^2(H_-, d\rho)$. The measure ρ on H_- is called the spectral measure of J. Note that the Jacobi field with

the Gaussian spectral measure is the classical free field in quantum field theory. The Jacobi field with the Poisson spectral measure was actually discovered in [15] and [25].

Jacobi fields are actively utilized in non-Gaussian white noise calculus and the theory of stochastic processes, see [7], [17], [2], [4], [5], [16], [11], [8], [9], [18], [19], [21], [20], and also [22] and [24]. Other applications are to the integration of nonlinear difference-differential equations, see [1]. In the case of a finite-dimensional H, the theory of Jacobi fields is closely related to some results in [13], [14], and [12].

The problem of finding an operator family with a given spectral measure often arises in applications. In some situations, the given measure is equal to the image of the spectral measure of a Jacobi field under a certain mapping. More precisely, let ρ be the spectral measure of the field J. Consider a mapping $K^+ : H_- \to T_-$ with T_- being a certain Hilbert space. This mapping takes ρ to the measure ρ_K on T_-. Our paper aims to find a family J_K of operators whose spectral measure equals ρ_K. In other words, we track the changes of the Jacobi field caused by mapping its spectral measure. Noteworthily, if K^+ is an invertible operator, then J_K appears to be isomorphic to the initial family J.

We also study the chaotic decomposition of the space $L^2(T_-, d\rho_K)$, which is derived through the orthogonalization of polynomials on T_-.

Throughout this paper, we assume K^+ to be a bounded operator with $\mathrm{Ker}(K^+) = \{0\}$. We will also assume $\mathrm{Ran}(K^+)$ to be dense in T_-. This assumption is not essential because the measure ρ_K is lumped on $\mathrm{Ran}(K^+)$, and we can always replace T_- with the closure of $\mathrm{Ran}(K^+)$ in T_-.

2 Preliminaries

Let H be a real separable Hilbert space. The corresponding symmetric Fock space is defined as

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H)$$

and consists of sequences $\Phi = (\Phi_n)_{n=0}^{\infty}$, $\Phi_n \in \mathcal{F}_n(H) = H_c^{\hat{\otimes} n}$, ($H_c$ being the complexification of H and $\hat{\otimes}$ denoting symmetric tensor product). The finite vectors $\Phi = (\Phi_1, \ldots, \Phi_n, 0, 0, \ldots) \in \mathcal{F}(H)$ form a linear topological space $\mathcal{F}_{fin}(H) \subset \mathcal{F}(H)$. The convergence in $\mathcal{F}_{fin}(H)$ is equivalent to the uniform finiteness and coordinate-wise convergence. The vector $\Omega = (1, 0, 0, \ldots) \in \mathcal{F}_{fin}(H)$ is called vacuum.

Let

$$H_- \supset H \supset H_+ \tag{2.1}$$

be a rigging of H with real Hilbert spaces H_+ and $H_- = (H_+)'$ (hereafter, X' denotes the dual of the space X). We suppose the inequality $\|\cdot\|_{H_+} \geq \|\cdot\|_H$ to hold for the norms. We also suppose the embedding $H_+ \hookrightarrow H$ to be quasinuclear. The pairing in (2.1) can be extended naturally to a pairing between $\mathcal{F}_n(H_+)$ and $\mathcal{F}_n(H_-)$. The latter can, in turn, be extended to a pairing between $\mathcal{F}_{fin}(H_+)$ and $(\mathcal{F}_{fin}(H_+))'$. In what follows, we use the notation $\langle \cdot, \cdot \rangle_H$ for all of these pairings. Note that $(\mathcal{F}_{fin}(H_+))'$ coincides with the direct product of the spaces $\mathcal{F}_n(H_-)$, $n \in \mathbb{Z}_+$.

2.1 Definition of a Jacobi field

In the Fock space $\mathcal{F}(H)$, consider the family $(\mathcal{J}(\phi))_{\phi \in H_+}$ of operator-valued Jacobi matrices

$$
\mathcal{J}(\phi) = \begin{pmatrix}
b_0(\phi) & a_0^*(\phi) & 0 & 0 & 0 & \cdots \\
a_0(\phi) & b_1(\phi) & a_1^*(\phi) & 0 & 0 & \cdots \\
0 & a_1(\phi) & b_2(\phi) & a_2^*(\phi) & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

with the entries

$$
a_n(\phi) : \mathrm{Dom}(a_n(\phi)) \to \mathcal{F}_{n+1}(H),
$$
$$
b_n(\phi) = (b_n(\phi))^* : \mathrm{Dom}(b_n(\phi)) \to \mathcal{F}_n(H),
$$
$$
a_n^*(\phi) = (a_n(\phi))^* : \mathrm{Dom}(a_n^*(\phi)) \to \mathcal{F}_n(H),
$$
$$
\phi \in H_+, \ n \in \mathbb{Z}_+ = 0, 1, \ldots.
$$

The inclusions $\mathrm{Dom}(a_n(\phi)) \subset \mathcal{F}_n(H)$, $\mathrm{Dom}(b_n(\phi)) \subset \mathcal{F}_n(H)$, and $\mathrm{Dom}(a_n^*(\phi)) \subset \mathcal{F}_{n+1}(H)$ hold for the domains. We suppose these domains to contain $\mathcal{F}_n(H_+)$ and $\mathcal{F}_{n+1}(H_+)$, respectively.

Each matrix $\mathcal{J}(\phi)$ gives rise to a Hermitian operator $J(\phi)$ in the space $\mathcal{F}(H)$: for $\Phi = (\Phi_n)_{n=0}^\infty \in \mathrm{Dom}(J(\phi)) = \mathcal{F}_{fin}(H_+)$ we define

$$
(J(\phi)\Phi)_n = a_{n-1}(\phi)\Phi_{n-1} + b_n(\phi)\Phi_n + a_n^*(\phi)\Phi_{n+1}, \qquad n \in \mathbb{Z}_+,
$$
$$
a_{-1}(\phi) = 0.
$$

Assume the following.

(a) The operators $a_n(\phi)$ and $b_n(\phi)$, $\phi \in H_+$, $n \in \mathbb{Z}_+$, take real spaces into real ones.

(b) (smoothness) The restrictions $a_n(\phi) \upharpoonright \mathcal{F}_n(H_+)$ and $b_n(\phi) \upharpoonright \mathcal{F}_n(H_+)$ act continuously from $\mathcal{F}_n(H_+)$ to $\mathcal{F}_{n+1}(H_+)$ and $\mathcal{F}_n(H_+)$, respectively. The restrictions $a_n^*(\phi) \upharpoonright \mathcal{F}_{n+1}(H_+)$ act continuously from $\mathcal{F}_{n+1}(H_+)$ to $\mathcal{F}_n(H_+)$.

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(c) The operators $J(\phi)$, $\phi \in H_+$, are essentially selfadjoint and their closures $\tilde{J}(\phi)$, $\phi \in H_+$, are strong commuting.

(d) The functions

$$H_+ \ni \phi \mapsto a_n(\phi)\Phi_n \in \mathcal{F}_{n+1}(H_+), \ \ H_+ \ni \phi \mapsto b_n(\phi)\Phi_n \in \mathcal{F}_n(H_+),$$
$$H_+ \ni \phi \mapsto a_n^*(\phi)\Phi_{n+1} \in \mathcal{F}_n(H_+), \qquad n \in \mathbb{Z}_+,$$

are linear and continuous for all $\Phi_n \in \mathcal{F}_n(H_+)$, $\Phi_{n+1} \in \mathcal{F}_{n+1}(H_+)$.

(e) (regularity) The real linear operators $V_n : \mathcal{F}_n(H_+) \to \bigoplus_{j=0}^n \mathcal{F}_j(H_+)$ defined by the equalities

$$V_0 = \mathrm{Id}_{\mathbb{C}}, \ \ V_n(\phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n) = J(\phi_1) \ldots J(\phi_n)\Omega,$$
$$\phi_1, \ldots, \phi_n \in H_+, \ n \in \mathbb{N},$$

are continuous. Furthermore, the operators

$$\mathcal{F}_n(H_+) \ni F_n \mapsto V_{n,n}F_n = (V_nF_n)_n \in \mathcal{F}_n(H_+), \qquad n \in \mathbb{N},$$

are invertible.

We will call the family $J = (\tilde{J}(\phi))_{\phi \in H_+}$ of operators a (commutative) Jacobi field if conditions (a)–(e) are satisfied. Once again we should emphasize that the operators $\tilde{J}(\phi)$ act in the Fock space $\mathcal{F}(H)$.

2.2 Spectral theory of a Jacobi field

It is possible to apply the projection spectral theorem, see [6] and [23], to the field $J = (\tilde{J}(\phi))_{\phi \in H_+}$. Here, we will only present the result of such an application.

Theorem 2.1. *Given a Jacobi field J, there exist a Borel probability measure ρ on the space H_- (the spectral measure) and a vector-valued function $H_- \ni \xi \mapsto P(\xi) \in (\mathcal{F}_{\mathrm{fin}}(H_+))'$ such that the following statements hold:*

1. *For every $\xi \in H_-$, the vector $P(\xi) = (P_n(\xi))_{n=0}^{\infty} \in (\mathcal{F}_{\mathrm{fin}}(H_+))'$, is a generalized joint eigenvector of J with eigenvalue ξ, i.e.,*

$$\langle P(\xi), \tilde{J}(\phi)\Phi \rangle_H = \langle \xi, \phi \rangle_H \langle P(\xi), \Phi \rangle_H, \qquad \phi \in H_+, \ \Phi \in \mathcal{F}_{\mathrm{fin}}(H_+). \qquad (2.2)$$

2. *After being extended by continuity to the whole of the space $\mathcal{F}(H)$, the Fourier transform*

$$\mathcal{F}(H) \supset \mathcal{F}_{\mathrm{fin}}(H_+) \ni \Phi = (\Phi_n)_{n=0}^{\infty} \mapsto (I\Phi)(\xi) = \langle \Phi, P(\xi) \rangle_H$$
$$= \sum_{n=0}^{\infty} \langle \Phi_n, P_n(\xi) \rangle_H \in L^2(H_-, d\rho) \qquad (2.3)$$

becomes a unitary operator between $\mathcal{F}(H)$ and $L^2(H_-, d\rho)$.

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3. *The mapping I takes every operator $\tilde{J}(\phi)$, $\phi \in H_+$, to the operator of multi-plication by the function $H_- \ni \xi \mapsto \langle \xi, \phi \rangle_H \in \mathbb{R}$ in the space $L^2(H_-, d\rho)$.*

Remark 2.1. The equality

$$IV_n F_n = \langle \xi^{\otimes n}, F_n \rangle_H, \qquad F_n \in \mathcal{F}_n(H_+), \ n \in \mathbb{Z}_+, \tag{2.4}$$

holds true. Indeed, Assertion 3 of Theorem 2.1 implies (2.4) for the vectors

$$\sigma_n = \sum_{k=1}^{l} \lambda_k \phi_{1,k} \hat{\otimes} \cdots \hat{\otimes} \phi_{n,k} \in \mathcal{F}_n(H_+),$$

$$\lambda_k \in \mathbb{C}, \ \phi_{i,k} \in H_+, \ i = 1, \ldots, n, \ l \in \mathbb{N}.$$

If a sequence $(\sigma_n^i)_{i=0}^\infty$ of such vectors converges to F_n in the space $\mathcal{F}_n(H_+)$, then

$$\langle \xi^{\otimes n}, \sigma_n^i \rangle_H = IV_n \sigma_n^i \to IV_n F_n$$

in the space $L^2(H_-, d\rho)$. Since $\langle \xi^{\otimes n}, \sigma_n^i \rangle_H \to \langle \xi^{\otimes n}, F_n \rangle_H$ for each $\xi \in H_-$, the above formula implies $IV_n F_n = \langle \xi^{\otimes n}, F_n \rangle_H$.

Now we have to recall some additional facts about the Fourier transform I.

Let $\mathcal{P}_n(H_-)$ denote the set of all continuous polynomials on H_- of degree $\leq n$:

$$H_- \ni \xi \mapsto p_n(\xi) = \sum_{j=0}^{n} \langle \xi^{\otimes j}, a_j \rangle_H \in \mathbb{C}, \qquad a_j \in \mathcal{F}_j(H_+), \ n \in \mathbb{Z}_+.$$

Theorem 2.2. *The Fourier transform I takes the set $\bigoplus_{j=0}^{n} \mathcal{F}_j(H_+) \subset \mathcal{F}(H)$, $n \in \mathbb{Z}_+$, to the set $\mathcal{P}_n(H_-) \subset L^2(H_-, d\rho)$ of continuous polynomials on H_- of degree $\leq n$, i.e.,*

$$I \left(\bigoplus_{j=0}^{n} \mathcal{F}_j(H_+) \right) = \mathcal{P}_n(H_-), \qquad n \in \mathbb{Z}_+.$$

The set $\mathcal{P}(H_-) = \bigcup_{n=0}^{\infty} \mathcal{P}_n(H_-)$ of all continuous polynomials on H_- is dense in $L^2(H_-, d\rho)$.

If $\dim H = \infty$, then $\bigoplus_{j=0}^{n} \mathcal{F}_j(H_+)$ is not closed in $\mathcal{F}(H)$ and neither is $\mathcal{P}_n(H_-)$ closed in $L^2(H_-, d\rho)$. The closure of $\mathcal{P}_n(H_-)$ in $L^2(H_-, d\rho)$ will be denoted by $\tilde{\mathcal{P}}_n(H_-)$. The elements of $\tilde{\mathcal{P}}_n(H_-)$ are, by definition, ordinary polynomials on H_-. Clearly,

$$I \left(\bigoplus_{j=0}^{n} \mathcal{F}_j(H) \right) = \tilde{\mathcal{P}}_n(H_-), \qquad n \in \mathbb{Z}_+.$$

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The orthogonal decomposition $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H)$ and the unitarity of I imply the following orthogonal (chaotic) decomposition of the space $L^2(H_-, d\rho)$:

$$L^2(H_-, d\rho) = \bigoplus_{n=0}^{\infty} (L_n^2),$$

$$(L_0^2) = \mathbb{C}, \ (L_n^2) = I(\mathcal{F}_n(H)) = \tilde{\mathcal{P}}_n(H_-) \ominus \tilde{\mathcal{P}}_{n-1}(H_-), \qquad n \in \mathbb{N}. \qquad (2.5)$$

Remark 2.2. Suppose $\mathcal{H}$ to be a nuclear space densely and continuously embedded into H. In all the previous constructions, it is possible to use the rigging

$$\mathcal{H}' \supset H \supset \mathcal{H}$$

instead of the rigging (2.1). In this case, the family J consists of the operators $\tilde{J}(\phi)$, $\phi \in \mathcal{H}$. The corresponding spectral measure ρ is a Borel probability measure on $\mathcal{H}'$.

2.3 Mapping of the spectral measure

Consider a real separable Hilbert space T_+ and a rigging

$$T_- \supset T_0 \supset T_+. \qquad (2.6)$$

As in the case of the rigging (2.1), the pairing in (2.6) can be extended to a pairing between $\mathcal{F}_n(T_+)$ and $\mathcal{F}_n(T_-)$. The latter can, in turn, be extended to a pairing between $\mathcal{F}_{fin}(T_+)$ and $(\mathcal{F}_{fin}(T_+))'$. We use the notation $\langle \cdot, \cdot \rangle_T$ for all of these pairings.

Let $K : T_+ \to H_+$ be a linear continuous operator with $\mathrm{Ker}(K) = \{0\}$ and suppose $\mathrm{Ran}(K)$ to be dense in H_+. The adjoint of K with respect to (2.1) and (2.6) is a linear continuous operator $K^+ : H_- \to T_-$ defined by the equality

$$\langle K^+ \xi, f \rangle_T = \langle \xi, K f \rangle_H, \qquad \xi \in H_-, \ f \in T_+.$$

Lemma 2.1. *The kernel* $\mathrm{Ker}(K^+) = \{0\}$. *The range* $\mathrm{Ran}(K^+)$ *is dense in* T_-.

Proof. Suppose $K^+ \xi = 0$ for some $\xi \in H_-$. This means $\langle K^+ \xi, f \rangle_T = \langle \xi, K f \rangle_H = 0$ for all $f \in T_+$. Since $\mathrm{Ran}(K)$ is dense in H_+, the latter implies $\xi = 0$. Thus $\mathrm{Ker}(K^+) = \{0\}$.

Next, we introduce a standard unitary $\mathbf{I}_T : T_- \to T_+$ by the formula

$$(\mathbf{I}_T \omega, f)_{T_+} = \langle \omega, f \rangle_T, \qquad \omega \in T_-, \ f \in T_+.$$

The equality

$$(K^+ \xi, \chi)_{T_-} = \langle K^+ \xi, \mathbf{I}_T \chi \rangle_T = \langle \xi, K \mathbf{I}_T \chi \rangle_H, \qquad \xi \in H_-, \ \chi \in T_-,$$

holds true. If $(K^+ \xi, \chi)_{T_-} = 0$ for any $\xi \in H_-$, then $K \mathbf{I}_T \chi = 0$ and $\chi = 0$. Thus $\mathrm{Ran}(K^+)$ is dense in T_-. $\qquad \square$

Let $\mathcal{B}(H_-)$ stand for the Borel σ-algebra of the space H_-. We denote by ρ_K the image of the measure ρ under the mapping K^+. By definition, ρ_K is a probability measure on the σ-algebra

$$\mathcal{C} = \{\Delta \subset T_- | (K^+)^{-1}(\Delta) \in \mathcal{B}(H_-)\},$$

$((K^+)^{-1}(\Delta)$ denoting the preimage of the set Δ). Clearly, the mapping K^+ is Borel-measurable, therefore $\mathcal{C}$ contains the Borel σ-algebra of the space T_-. If K^+ takes Borel subsets of H_- to the Borel subsets of T_-, then $\mathcal{C}$ coincides with the Borel σ-algebra of T_-.

3 Main results

Consider a Jacobi field $J = (\tilde{J}(\phi))_{\phi \in H_+}$ in the Fock space $\mathcal{F}(H)$. The spectral measure ρ of the field J is defined on H_-. The mapping K^+ takes ρ to the measure ρ_K on T_-. The main objectives of this section are:

1. To obtain a family $J_K = (\tilde{J}_K(f))_{f \in T_+}$ of commuting selfadjoint operators operators whose spectral measure is equal to ρ_K.

2. To obtain an analogue of the decomposition (2.5) for the space $L^2(T_-, d\rho_K)$.

We note that the family J_K proves to satisfy conditions (a)–(d) of a Jacobi field. It is generally unclear whether J_K satisfies condition (e).

The assumption $\mathrm{Ker}(K) = \{0\}$ is not essential. Indeed, the measure ρ_K proves to be lumped on the set of functionals which equal zero on $\mathrm{Ker}(K)$. This set can be naturally identified with $(\mathrm{Ker}(K)^\perp)'$. Thus we can always replace T_+ with $\mathrm{Ker}(K)^\perp \subset T_+$.

3.1 ρ_K as the spectral measure

Define the Hilbert space T as the completion of T_+ with respect to the scalar product

$$(f_1, f_2)_T = (Kf_1, Kf_2)_H, \qquad f_1, f_2 \in T_+.$$

The operator K induces a unitary $\bar{K} : T \to H$. We preserve the notations K and $\bar{K}$ for the extensions of K and $\bar{K}$ to the complexified spaces $T_{+,c}$ and T_c.

In the Fock space $\mathcal{F}(T) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(T)$, consider the family $(\mathcal{J}_K(f))_{f \in T_+}$ of operator-valued Jacobi matrices

$$\mathcal{J}_K(f) = \begin{pmatrix} \beta_0(f) & \alpha_0^*(f) & 0 & 0 & 0 & \cdots \\ \alpha_0(f) & \beta_1(f) & \alpha_1^*(f) & 0 & 0 & \cdots \\ 0 & \alpha_1(f) & \beta_2(f) & \alpha_2^*(f) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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with the entries

$$\alpha_n(f) = (\bar{K}^{\otimes(n+1)})^{-1} a_n(Kf)\bar{K}^{\otimes n} : \mathrm{Dom}(\alpha_n(f)) \to \mathcal{F}_{n+1}(T),$$
$$\beta_n(f) = (\bar{K}^{\otimes n})^{-1} b_n(Kf)\bar{K}^{\otimes n} : \mathrm{Dom}(\beta_n(f)) \to \mathcal{F}_n(T),$$
$$\alpha_n^*(f) = (\alpha_n(f))^* : \mathrm{Dom}(\alpha_n^*(f)) \to \mathcal{F}_n(T),$$
$$f \in T_+, \ n \in \mathbb{Z}_+,$$

(recall that $a_n(\phi)$ and $b_n(\phi)$ denote the entries of $\mathcal{J}(\phi)$). The domains $\mathrm{Dom}(\alpha_n(f))$, $\mathrm{Dom}(\beta_n(f))$, and $\mathrm{Dom}(\alpha_n^*(f))$ contain $\mathcal{F}_n(T_+)$ and $\mathcal{F}_{n+1}(T_+)$, respectively. As in the case of $\mathcal{J}(\phi)$, each matrix $\mathcal{J}_K(f)$ gives rise to a Hermitian operator $J_K(f)$ in the space $\mathcal{F}(T)$. The domain $\mathrm{Dom}(J_K(f))$ equals $\mathcal{F}_{fin}(T_+)$.

As we will further show, the operators $J_K(f)$, $f \in T_+$, are essentially selfadjoint in the space $\mathcal{F}(T)$. Their closures $\tilde{J}_K(f)$ are strongly commuting. Denote $J_K = (\tilde{J}_K(f))_{f \in T_+}$.

Theorem 3.1. *Assume the restrictions*

$$a_n(Kf) \upharpoonright (\mathrm{Ran}(K))^{\hat{\otimes}n}, \ b_n(Kf) \upharpoonright (\mathrm{Ran}(K))^{\hat{\otimes}n},$$
$$a_n^*(Kf) \upharpoonright (\mathrm{Ran}(K))^{\hat{\otimes}(n+1)}, \qquad f \in T_+, \ n \in \mathbb{Z}_+, \quad (3.1)$$

to take values in $(\mathrm{Ran}(K))^{\hat{\otimes}(n+1)}$ and $(\mathrm{Ran}(K))^{\hat{\otimes}n}$, respectively. There exists a vector-valued function $T_- \ni \omega \mapsto Q(\omega) \in (\mathcal{F}_{\mathrm{fin}}(T_+))'$ such that the following statements hold:

1. *For ρ_K-almost all $\omega \in T_-$, the vector $Q(\omega) = (Q_n(\omega))_{n=0}^\infty \in (\mathcal{F}_{\mathrm{fin}}(T_+))'$, is a generalized joint eigenvector of the family J_K with the eigenvalue ω, i.e.,*

$$\langle Q(\omega), \tilde{J}_K(f)F \rangle_T = \langle \omega, f \rangle_T \langle Q(\omega), F \rangle_T, \qquad F \in \mathcal{F}_{\mathrm{fin}}(T_+). \quad (3.2)$$

2. *After being extended by continuity to the whole of the space $\mathcal{F}(T)$, the Fourier transform*

$$\mathcal{F}(T) \supset \mathcal{F}_{\mathrm{fin}}(T_+) \ni F = (F_n)_{n=0}^\infty \mapsto (I_K F)(\omega) = \langle F, Q(\omega) \rangle_T$$
$$= \sum_{n=0}^\infty \langle F_n, Q_n(\omega) \rangle_T \in L^2(T_-, d\rho_K) \quad (3.3)$$

becomes a unitary between $\mathcal{F}(T)$ and $L^2(T_-, d\rho_K)$.

3. *The mapping I_K takes every operator $\tilde{J}_K(f)$, $f \in T_+$, to the operator of multiplication by the function $T_- \ni \omega \mapsto \langle \omega, f \rangle_T \in \mathbb{R}$ in the space $L^2(T_-, d\rho_K)$.*

Proof. **Step 1.** First, we have to prove that the operators $J_K(f)$, $f \in T_+$, are essentially selfadjoint and their closures are strongly commuting. We define the operator

$$\mathcal{K} = \bigoplus_{n=0}^{\infty} \bar{K}^{\otimes n} : \mathcal{F}(T) \to \mathcal{F}(H). \tag{3.4}$$

The unitarity of $\bar{K}$ implies the unitarity of $\mathcal{K}$. A straightforward calculation shows that

$$J_K(f) = \mathcal{K}^{-1} J(Kf) \mathcal{K}, \qquad f \in T_+. \tag{3.5}$$

The operators $J(Kf)$ are essentially selfadjoint and their closures are strongly commuting. Since $\mathcal{K}$ is a unitary, the operators $J_K(f)$ possess these properties, too.

Step 2. Let us establish an isomorphism between the spaces $L^2(T_-, d\rho_K)$ and $L^2(H_-, d\rho)$. For a complex-valued function $G(\omega)$ on T_-, we define the function

$$(\mathcal{U}G)(\xi) = G(K^+\xi), \qquad \xi \in H_-. \tag{3.6}$$

According to the definition of ρ_K, the mapping $\mathcal{U}$ induces an isometric operator U between the spaces $L^2(T_-, d\rho_K)$ and $L^2(H_-, d\rho)$.

We have $\mathrm{Ran}(U) = L^2(H_-, d\rho)$. Indeed, consider an arbitrary function $F(\xi)$ over H_-. Define $G(\omega) = F((K^+)^{-1}\omega)$ if $\omega \in \mathrm{Ran}(K^+)$ and $G(\omega) = 0$ otherwise. The equality

$$(\mathcal{U}G)(\xi) = G(K^+\xi) = F(\xi), \qquad \xi \in H_-,$$

holds true. If $F \in L^2(H_-, d\rho)$, then $G \in L^2(T_-, d\rho_K)$. In this case, the above equality yields $(UG)(\xi) = F(\xi)$.

As a result, we have the unitary $U : L^2(T_-, d\rho_K) \to L^2(H_-, d\rho)$.

Step 3. Consider the operator

$$I_K = U^{-1} I \mathcal{K} : \mathcal{F}(T) \to L^2(T_-, d\rho_K)$$

with I and $\mathcal{K}$ given by (2.3) and (3.4), respectively. Since all of its components are unitaries between the corresponding spaces, I_K is a unitary itself. Our next goal is to establish representation (3.3) for I_K.

Fix a vector $F \in \mathcal{F}_{\mathrm{fin}}(T_+)$. According to (2.3), the equality

$$(I\mathcal{K}F)(\xi) = \langle \mathcal{K}F, P(\xi) \rangle_H = \sum_{n=0}^{\infty} \langle F, (K^+)^{\otimes n} P_n(\xi) \rangle_H, \qquad \xi \in H_-,$$

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holds true. Define $Q(K^+\xi) = ((K^+)^{\otimes n} P_n(\xi))_{n=0}^{\infty} \in (\mathcal{F}_{\text{fin}}(T_+))'$. Note that $Q(K^+\xi)$ is well-defined because K^+ is monomorphic. Evidently,

$$(I\mathcal{K}F)(\xi) = \langle F, Q(K^+\xi)\rangle_T, \qquad \xi \in H_-. \tag{3.7}$$

The application of U^{-1} to (3.7) yields representation (3.3) for the unitary $I_K = U^{-1}I\mathcal{K}$.

The proof of Theorem 3.1 will be complete if we show that Statements 1 and 3 hold for the function $Q(\omega)$.

Step 4. Let us prove Statement 1. As before, we fix a vector $F \in \mathcal{F}_{\text{fin}}(T_+)$. Due to the assumption on the restrictions (3.1), the vector $\tilde{J}_K(f)F$ belongs to $\mathcal{F}_{\text{fin}}(T_+)$. Hence the right-hand side of (3.2) is well-defined.

Formulas (2.2), (3.5), and (3.7) imply

$$
\begin{aligned}
\langle Q(\omega), \tilde{J}_K(f)F\rangle_T &= \overline{\langle \tilde{J}_K(f)F, Q(\omega)\rangle_T} \\
&= \overline{(I_K \tilde{J}_K(f)F)(\omega)} \\
&= \overline{(U^{-1}I\mathcal{K}\tilde{J}_K(f)F)(\omega)} \\
&= \overline{(U^{-1}I\mathcal{K}\mathcal{K}^{-1}\tilde{J}(Kf)\mathcal{K}F)(\omega)} \\
&= \overline{(U^{-1}I\tilde{J}(Kf)\mathcal{K}F)(\omega)} \\
&= (U^{-1}\overline{\langle \tilde{J}(Kf)\mathcal{K}F, P(\cdot)\rangle_H})(\omega) \\
&= (U^{-1}(\overline{\langle Kf, \cdot\rangle_H \langle \mathcal{K}F, P(\cdot)\rangle_H}))(\omega) \\
&= (U^{-1}\overline{\langle Kf, \cdot\rangle_H})(\omega)(U^{-1}\overline{\langle \mathcal{K}F, P(\cdot)\rangle_H})(\omega) \\
&= \langle f, \omega\rangle_T \overline{(U^{-1}I\mathcal{K}F)(\omega)} \\
&= \langle f, \omega\rangle_T \overline{(I_K F)(\omega)} \\
&= \langle f, \omega\rangle_T \langle Q(\omega), F\rangle_T
\end{aligned}
$$

for ρ_K-almost all $\omega \in T_-$ (overbars denote complex conjugacy). This proves Statement 1.

Statement 3 is a direct consequence of (3.3) and (3.2). $\qquad\square$

Remark 3.1. While proving the theorem, we showed that the mapping (3.6) induces a unitary $U : L^2(T_-, d\rho_K) \to L^2(H_-, d\rho)$. We also obtained an explicit formula for the Fourier transform I_K. Namely,

$$I_K = U^{-1}I\mathcal{K} \tag{3.8}$$

with I and $\mathcal{K}$ given by (2.3) and (3.4), respectively.

Remark 3.2. As mentioned above, it is generally unclear whether J_K satisfies condition (e) in the definition of a Jacobi field. However, if the operator K is invertible, then J_K does satisfy (e) and hence is a Jacobi field. This field is isomorphic to the initial field J.

3.2 Orthogonal (chaotic) decomposition of the space $L^2(T_-, d\rho_K)$

This subsection aims to obtain an analogue of the decomposition (2.5) for the space $L^2(T_-, d\rho_K)$. If J_K proves to be a Jacobi field, then Theorem 2.2 is applicable. Otherwise, an analogue of (2.5) for $L^2(T_-, d\rho_K)$ may be obtained with the help of Theorem 3.2.

Further considerations do not require any assumptions on the restrictions (3.1). Theorem 3.2 below is applicable to a Jacobi field which does not satisfy the assumption of Theorem 3.1. In this case, the unitary I_K should be defined by formula (3.8).

Let $\mathcal{Q}_n(T_-)$ denote the set of all continuous polynomials

$$T_- \ni \omega \mapsto q_n(\omega) = \sum_{j=0}^{n} \langle \omega^{\otimes j}, c_j \rangle_T \in \mathbb{C}, \qquad c_j \in \mathcal{F}_j(T_+), \ n \in \mathbb{Z}_+, \qquad (3.9)$$

on T_- of degree $\leq n$. As will be shown below, the inclusion

$$\mathcal{Q}_n(T_-) \subset L^2(T_-, d\rho_K) \qquad (3.10)$$

holds. The closure of $\mathcal{Q}_n(T_-)$ in $L^2(T_-, d\rho_K)$ will be denoted by $\tilde{\mathcal{Q}}_n(T_-)$. The elements of $\tilde{\mathcal{Q}}_n(T_-)$ are ordinary polynomials on T_-.

Theorem 3.2. *The unitary I_K takes the set $\bigoplus_{j=0}^{n} \mathcal{F}_j(T) \subset \mathcal{F}(T)$, $n \in \mathbb{Z}_+$, to the set $\tilde{\mathcal{Q}}_n(T_-) \subset L^2(T_-, d\rho_K)$ of ordinary polynomials on T_-, i.e.,*

$$I_K \left(\bigoplus_{j=0}^{n} \mathcal{F}_j(T) \right) = \tilde{\mathcal{Q}}_n(T_-), \qquad n \in \mathbb{Z}_+. \qquad (3.11)$$

The set $\mathcal{Q}(T_-) = \bigcup_{n=0}^{\infty} \mathcal{Q}_n(T_-)$ of all continuous polynomials on T_- is dense in $L^2(T_-, d\rho_K)$.

Proof. **Step 1.** First, we have to prove inclusion (3.10). The application of the mapping (3.6) to the polynomial (3.9) yields

$$(\mathcal{U}q_n)(\xi) = q_n(K^+\xi) = \sum_{j=0}^{n} \langle (K^+\xi)^{\otimes j}, c_j \rangle_T$$

$$= \sum_{j=0}^{n} \langle (K^+)^{\otimes j}\xi^{\otimes j}, c_j \rangle_T = \sum_{j=0}^{n} \langle \xi^{\otimes j}, K^{\otimes j}c_j \rangle_H.$$

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The expression in the right hand side of this formula is a continuous polynomial with the coefficients $a_j = K^{\otimes j} c_j \in \mathcal{F}_j(H_+)$. According to Theorem 2.2, this polynomial belongs to the space $L^2(H_-, d\rho)$. Therefore $q_n(\omega)$ belongs to the space $L^2(T_-, d\rho_K)$. The latter proves inclusion (3.10).

Step 2. Let us prove equality (3.11). Formula (3.8) and Theorem 2.2 yield

$$
\begin{aligned}
I_K \left(\bigoplus_{j=0}^{n} \mathcal{F}_j(T) \right) &= U^{-1} I \mathcal{K} \left(\bigoplus_{j=0}^{n} \mathcal{F}_j(T) \right) \\
&= U^{-1} I \left(\bigoplus_{j=0}^{n} \mathcal{F}_j(H) \right) = U^{-1} \tilde{\mathcal{P}}_n(H_-), \qquad n \in \mathbb{Z}_+.
\end{aligned}
$$

The proof of equality (3.11) will be complete if we show that $\tilde{\mathcal{P}}_n(H_-) = U \tilde{\mathcal{Q}}_n(T_-)$, $n \in \mathbb{Z}_+$.

As explained above, each function $U q_n(\xi) = q_n(K^+ \xi)$, $q_n(\omega) \in \mathcal{Q}_n(T_-)$, belongs to $\mathcal{P}_n(H_-)$. Thus it is only necessary to prove that such functions are dense in $\tilde{\mathcal{P}}_n(H_-)$.

Step 3. It suffices to approximate a monomial $\langle \xi^{\otimes m}, a_m \rangle_H$, $a_m \in \mathcal{F}_m(H_+)$, $m = 1, \ldots, n$, with the elements of $U \mathcal{Q}_n(T_-)$.

Fix $\epsilon > 0$. Since $\mathrm{Ran}(K)$ is dense in H_+, there exists a vector

$$
s_{m,\epsilon} = \sum_{k=1}^{l} \lambda_k f_{1,k} \hat{\otimes} \cdots \hat{\otimes} f_{m,k} \in \mathcal{F}_m(T_+),
$$

$$
\lambda_k \in \mathbb{C}, \ f_{i,k} \in T_+, \ i = 1, \ldots, m, \ l \in \mathbb{N},
$$

such that

$$
\| a_m - K^{\otimes n} s_{m,\epsilon} \|_{\mathcal{F}(H_+)} = \left\| a_m - \sum_{k=1}^{l} \lambda_k K f_{1,k} \hat{\otimes} \cdots \hat{\otimes} K f_{m,k} \right\|_{\mathcal{F}(H_+)} < \epsilon.
$$

Taking equality (2.4) into account, we conclude that the monomial $\langle \omega^{\otimes m}, s_{m,\epsilon} \rangle_T \in$

$\mathcal{Q}_n(T_-)$ satisfies the estimate

$$
\begin{aligned}
\|\langle \xi^{\otimes m}, a_m \rangle_H &- (U\langle \cdot^{\otimes m}, s_{m,\epsilon} \rangle_T)(\xi)\|_{L^2(H_-, d\rho(\xi))} \\
&= \|\langle \xi^{\otimes m}, a_m \rangle_H - \langle (K^+)^{\otimes m} \xi^{\otimes m}, s_{m,\epsilon} \rangle_T\|_{L^2(H_-, d\rho(\xi))} \\
&= \|\langle \xi^{\otimes m}, a_m \rangle_H - \langle \xi^{\otimes m}, K^{\otimes m} s_{m,\epsilon} \rangle_H\|_{L^2(H_-, d\rho(\xi))} \\
&= \|I^{-1}(\langle \xi^{\otimes m}, a_m \rangle_H - \langle \xi^{\otimes m}, K^{\otimes m} s_{m,\epsilon} \rangle_H)\|_{\mathcal{F}(H)} \\
&= \|I^{-1}(IV_m a_m - IV_m K^{\otimes m} s_{m,\epsilon})\|_{\mathcal{F}(H)} \\
&= \|V_m(a_m - K^{\otimes m} s_{m,\epsilon})\|_{\mathcal{F}(H)} \\
&\leq \|V_m(a_m - K^{\otimes m} s_{m,\epsilon})\|_{\mathcal{F}(H_+)} \\
&\leq \|V_m\| \|a_m - K^{\otimes m} s_{m,\epsilon}\|_{\mathcal{F}(H_+)} < \|V_m\|\epsilon.
\end{aligned}
$$

Thus we have approximated $\langle \xi^{\otimes m}, a_m \rangle_H$ with the functions $(U\langle \cdot^{\otimes m}, s_{m,\epsilon} \rangle_T)(\xi) \in U\mathcal{Q}_n(T_-)$.

Step 4. Let us prove the last assertion of Theorem 3.2. Due to the unitarity of I_K,

$$
\begin{aligned}
L^2(T_-, d\rho_K) = I_K(\mathcal{F}(T)) = (I_K(\mathcal{F}_{fin}(T)))^\sim &= \left(\bigcup_{n=0}^{\infty} I_K \left(\bigoplus_{m=0}^{n} \mathcal{F}_m(T) \right) \right)^\sim \\
&= \left(\bigcup_{n=0}^{\infty} \tilde{\mathcal{Q}}_n(T_-) \right)^\sim = \left(\bigcup_{n=0}^{\infty} \mathcal{Q}_n(T_-) \right)^\sim,
\end{aligned}
$$

(tilde stands for the closure in the corresponding space). Thus $\mathcal{Q}(T_-)$ is dense in $L^2(T_-, d\rho_K)$. $\qquad\square$

We can now construct the (2.5)-type decomposition for the space $L^2(T_-, d\rho_K)$:

$$
L^2(T_-, d\rho_K) = \bigoplus_{n=0}^{\infty} (L_n^2)_K,
$$

$$
(L_0^2)_K = \mathbb{C}, \ (L_n^2)_K = I_K(\mathcal{F}_n(T)) = \tilde{\mathcal{Q}}_n(T_-) \ominus \tilde{\mathcal{Q}}_{n-1}(T_-), \qquad n \in \mathbb{N}.
$$

4   Examples

Let us make some remarks concerning the space T and the Fourier transform of the measure ρ_K,

$$
\hat{\rho}_K(f) = \int_{T_-} e^{i\langle \omega, f \rangle_T} d\rho_K(\omega), \qquad f \in T_+.
$$

We will be using these remarks in our further considerations.

Remark 4.1. Since $K : T_+ \to H_+$ is continuous and since the embedding $H_+ \hookrightarrow H$ is continuous, we easily conclude that T_+ is continuously embedded into T. Furthermore, T_+ is a dense subset of T. Thus we can use T as the zero space in the chain (2.6), i.e., we can assume $T_0 = T$.

Remark 4.2. The set $\mathrm{Ran}(K) \subset H_+ \subset H_-$ is dense in H_-. Assuming $T_0 = T$, one can prove that the restriction $K^+ \upharpoonright \mathrm{Ran}(K) : \mathrm{Ran}(K) \to T_-$ coincides with the mapping $K^{-1} : \mathrm{Ran}(K) \to T_+ \subset T_-$.

Remark 4.3. Consider the Fourier transform

$$\hat{\rho}(\phi) = \int_{H_-} e^{i\langle \xi, \phi \rangle_H} d\rho(\xi), \qquad \phi \in H_+,$$

of the measure ρ. By the definition of ρ_K, we have:

$$\hat{\rho}_K(f) = \int_{T_-} e^{i\langle \omega, f \rangle_T} d\rho_K(\omega) = \int_{H_-} e^{i\langle K^+ \xi, f \rangle_H} d\rho(\xi)$$

$$= \int_{H_-} e^{i\langle \xi, Kf \rangle_H} d\rho(\xi) = \hat{\rho}(Kf), \qquad f \in T_+.$$

Thus the Fourier transform $\hat{\rho}_K(f)$ of the measure ρ_K satisfies the equality

$$\hat{\rho}_K(f) = \hat{\rho}(Kf), \qquad f \in T_+.$$

We will now apply the results of Section 3 to some classical Jacobi fields.

Example 4.1. Suppose J to be the classical free field, see e.g. [6], [7], [17], [2], and [3]. In this case,

$$a_n(\phi)\Phi_n = (\sqrt{n+1}\phi)\hat{\otimes}\Phi_n, \ b_n(\phi)\Phi_n = 0, \qquad \Phi_n \in \mathcal{F}_n(H), \ \phi \in H_+, \ n \in \mathbb{Z}_+.$$

Clearly, the assumption of Theorem 3.1 on the restrictions (3.1) is now automatically satisfied for any operator K under consideration.

The spectral measure ρ of the field J is the standard Gaussian measure γ on H_-. Its Fourier transform is given by the formula

$$\hat{\rho}(\phi) = \hat{\gamma}(\phi) = \exp\left(-\frac{1}{2}\|\phi\|_H^2\right), \qquad \phi \in H_+.$$

According to Remark 4.3, the Fourier transform of ρ_K is given by the formula

$$\hat{\rho}_K(f) = \hat{\rho}(Kf) = \exp\left(-\frac{1}{2}\|Kf\|_H^2\right) = \exp\left(-\frac{1}{2}\langle K^+Kf, f \rangle_T\right), \qquad f \in T_+,$$

(since H_+ is a subset of H_-, the operator $K^+K : T_+ \to T_-$ is well-defined). This means ρ_K is the Gaussian measure on T_- with the correlation operator K^+K. Notice

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that, in the case where $T_0 = T$, the Fourier transform of ρ_K may be written down in the form

$$\hat{\rho}_K(f) = \exp\left(-\frac{1}{2}\|f\|_T^2\right), \qquad f \in T_+,$$

i.e., ρ_K is the standard Gaussian measure on T_-.

Applying Theorem 3.1 to the classical free field J, we obtain the family J_K whose spectral measure is ρ_K.

In what follows, we assume $T_0 = T$.

Example 4.2. Let H be $L^2(\mathbb{R}, dx)$ and let H_+ and T_+ be the Sobolev spaces $W_2^1(\mathbb{R}, (1 + x^2)\,dx)$ and $W_2^1(\mathbb{R}, dx)$, respectively. Suppose J to be the Poisson field, see e.g. [17], [2], [3], and [5]. In this case,

$$\begin{aligned}
a_0(\phi)\Phi_0 &= \Phi_0\phi, \ b_0(\phi)\Phi_0 = 0, \\
a_n(\phi)\Phi_n &= (\sqrt{n+1}\phi)\hat{\otimes}\Phi_n, \\
b_n(\phi)\Phi_n &= (b(\phi) \otimes \mathrm{Id}_{H_+} \otimes \cdots \otimes \mathrm{Id}_{H_+})\Phi_n \\
&\quad + (\mathrm{Id}_{H_+} \otimes b(\phi) \otimes \mathrm{Id}_{H_+} \otimes \cdots \otimes \mathrm{Id}_{H_+})\Phi_n + \cdots \\
&\quad + (\mathrm{Id}_{H_+} \otimes \cdots \otimes \mathrm{Id}_{H_+} \otimes b(\phi))\Phi_n, \\
\Phi_0 &\in \mathcal{F}_0(H_+), \ \Phi_n \in \mathcal{F}_n(H_+), \ \phi \in H_+, \ n \in \mathbb{N}.
\end{aligned}$$

Here, $b(\phi)$ the operator of multiplication by the function $\phi(x)$ in the space H.

The space H_- coincides with the negative Sobolev space $W_2^{-1}(\mathbb{R}, (1 + x^2)\,dx)$. The spectral measure ρ of the field J is equal to the centered Poisson measure π with the intensity dx. The Fourier transform of ρ is given by the formula

$$\hat{\rho}(\phi) = \hat{\pi}(\phi) = \exp\left(\int_{\mathbb{R}} (e^{i\phi(x)} - 1 - i\phi(x))\,dx\right), \qquad \phi \in H_+.$$

Suppose $K : T_+ \to H_+$ to be the operator of multiplication by the function $\kappa(x) = e^{-x^2}$. One can easily verify that K is bounded and $\mathrm{Ker}(K) = \{0\}$. The range $\mathrm{Ran}(K)$ is dense in H_+ because it contains all the smooth compactly supported functions. On the other hand, $\mathrm{Ran}(K) \neq H_+$ because e.g. the function $\psi(x) = (1 + x^2)^{-2} \in H_+$ does not belong to $\mathrm{Ran}(K)$. Clearly, the field J and the operator K satisfy the assumptions of Theorem 3.1.

The space T_- is the dual of the space $W_2^1(\mathbb{R}, dx)$ with respect to the zero space $T = L^2(\mathbb{R}, e^{-2x^2}\,dx)$. Evidently, one may realize T_- as the dual space of $W_2^1(\mathbb{R}, dx)$ with respect to the zero space $L^2(\mathbb{R}, dx)$, in which case T_- is the usual negative Sobolev space $W_2^{-1}(\mathbb{R}, dx)$.

According to Remark 4.2, the operator $K^+ : H_- \to T_-$ is equal to the extension by continuity of the mapping

$$H_- \supset \mathrm{Ran}(K) \ni \xi(x) \mapsto e^{x^2}\xi(x) \in T_-.$$

According to Remark 4.3, the Fourier transform of ρ_K is given by the formula

$$\hat{\rho}_K(f(x)) = \hat{\rho}(e^{-x^2}f(x))$$

$$= \exp\left(\int_{\mathbb{R}} \left(e^{ie^{-x^2}f(x)} - 1 - ie^{-x^2}f(x)\right) dx\right), \qquad f \in T_+.$$

Applying Theorem 3.1, we obtain the family J_K whose spectral measure is ρ_K.

Example 4.3. As before, let H be $L^2(\mathbb{R}, dx)$. Let H_+ and T_+ equal $W_2^1\left(\mathbb{R}, e^{\frac{x^2}{2}} dx\right)$ and $W_2^2\left(\mathbb{R}, e^{\frac{x^2}{2}} dx\right)$, respectively. Suppose J to be the Poisson field.

Define the operator $K : T_+ \to H_+$ as the extension by continuity of the mapping

$$C_0^\infty(\mathbb{R}) \ni p(x) \mapsto e^{-\frac{x^2}{2}}\frac{dp(x)}{dx} \in H_+$$

($C_0^\infty(\mathbb{R})$ stands for the set of all smooth compactly supported functions on $\mathbb{R}$). Evidently, K is bounded and $\text{Ker}(K) = \{0\}$.

Lemma 4.1. *The range* $\text{Ran}(K)$ *is dense in* H_+.

Proof. Fix $q \in H_+$ and assume $(Kp(x), q(x))_{H_+} = 0$ for an arbitrary $p \in C_0^\infty(\mathbb{R})$. Our goal is to show that $q = 0$.

The equality

$$(Kp, q)_{H_+} = \int_{\mathbb{R}} \left(-\frac{d^3 p(x)}{dx^3} + x\frac{d^2 p(x)}{dx^2} + 2\frac{dp(x)}{dx}\right) q(x)\, dx$$

holds. Consider the differential expression

$$\mathcal{L} = -\frac{d^3}{dx^3} + x\frac{d^2}{dx^2} + 2\frac{d}{dx}.$$

Let $\mathcal{L}^+$ denote the adjoint expression. Since

$$(Kp, q)_{H_+} = (\mathcal{L}p, q)_H = 0$$

for an arbitrary $p \in C_0^\infty(\mathbb{R})$, the function q is a generalized solution of the differential equation $\mathcal{L}^+ y = 0$. Calculating $\mathcal{L}^+$ and applying Theorem 6.1 from Chapter 16 of [10], we conclude that q is indeed a classical solution of the equation

$$\frac{d^3 y(x)}{dx^3} + x\frac{d^2 y(x)}{dx^2} = 0.$$

The general solution of the above equation is

$$y(x) = c_1 \int_0^x \int_0^t e^{-\frac{s^2}{2}}\, ds\, dt + c_2 x + c_3, \qquad c_1, c_2, c_3 \in \mathbb{C}.$$

Assume $y = q \in H_+$. In this case, the limits

$$\lim_{x \to \infty} \frac{dy(x)}{dx} = c_1 \lim_{x \to \infty} \int_0^x e^{-\frac{t^2}{2}} dt + c_2 = c_1 \sqrt{\frac{\pi}{2}} + c_2,$$

$$\lim_{x \to -\infty} \frac{dy(x)}{dx} = -c_1 \lim_{x \to -\infty} \int_x^0 e^{-\frac{t^2}{2}} dt + c_2 = -c_1 \sqrt{\frac{\pi}{2}} + c_2$$

must equal 0. Evidently, the latter implies $c_1 = c_2 = c_3 = 0$. Thus $q = 0$. $\square$

The Poisson field J and the operator K do not satisfy the assumptions of Theorem 3.1. However, Theorem 3.2 is applicable now.

The space H_- is the negative Sobolev space $W_2^{-1}\big(\mathbb{R}, e^{\frac{x^2}{2}} dx\big)$, while T_- may be realized as the dual of the space $W_2^2(\mathbb{R}, e^{\frac{x^2}{2}} dx)$ with respect to the zero space $L^2(\mathbb{R}, dx)$. In this case, T_- is the usual negative Sobolev space $W_2^{-2}(\mathbb{R}, e^{\frac{x^2}{2}} dx)$. According to Remark 4.3, the Fourier transform of ρ_K is given by the formula

$$\hat{\rho}_K(f(x)) = \hat{\rho}\left(e^{-\frac{x^2}{2}} \frac{df(x)}{dx}\right)$$

$$= \exp\left(\int_{\mathbb{R}} \left(\exp\left(ie^{-\frac{x^2}{2}} \frac{df(x)}{dx}\right) - 1 - ie^{-\frac{x^2}{2}} \frac{df(x)}{dx}\right) dx\right), \qquad f \in T_+.$$

Applying Theorem 3.2, we obtain a (2.5)-type decomposition for the space $L^2(T_-, d\rho_K)$.

In a forthcoming paper, we are going to discuss in detail the case of the fractional Brownian motion, which is an important example of a Gaussian measure with a non-trivial correlation operator.

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